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Equilibrium conditions and vector variational inequalities: a complex relation

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Abstract Since a look to the present literature shows that the notion of vector equilibrium is not univocally defined, the relationship between vector equilibrium problems and vector variational inequalities deserves a specific analysis, deeper than in the scalar case. We analyse and compare various definitions and clarify how they depend on the modelling of the problem under consideration. Furthermore, we show that at least one vector variational inequality formulation of the problem is possible, which is equivalent to a Wardrop-type equilibrium condition. The analysis is illustrated through the concrete example of a multiclass and multiobjective supply demand network.

Keywords Traffic Networks · Vector Variational Inequalities · Pareto Optimization · Multiobjective Equilibrium Problems · Vector Wardrop Principle

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1 Introduction

This note aims to point out the difference between the scalar and the vector equilibrium problems and the associated scalar and vector variational inequalities (VVI). As the impressive literature shows, e.g. [6], all the scalar equilibrium problems arising from various fields of applied sciences can be formulated in terms of a variational inequality (VI) over a certain convex set. From the VI one can also recover the original equilibrium problem (eventually, by adding some natural assumptions which can depend on the particular problem under consideration). On the contrary, in the vector case (see e.g. [5,7,10,11] for applications of VVI to equilibrium problems) the relationship between equilibrium problems and VVI is not univocally determined, as a consequence of the fact that different notions of equilibrium can

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be given. In general, the proposed equilibrium conditions are *not* connected, in an equivalent way, with a VVI and, hence, the new problem arises to see wether there are conditions which lead to an equivalent VVI formulation.

In order to clarify the above claims, we study the vector traffic equilibrium problem, which presents all the features considered. Of course, the discussion that we shall perform in this case holds for other kinds of equilibrium problems, because these problems can have a unified formulation and, as is well known, in the scalar case, the equilibrium conditions derived from Wardrop's principle are equivalently expressed by a variational inequality (see for instance [2,4,9]).

The vector nature of equilibrium problems arises, in the most general case, from two different modelling requirements: on the one hand when we search an equilibrium with respect to several objectives (thus generalizing Pareto's optimization); on the other hand, when we want to group the variables in different classes so that a vector partial order appears also in the feasible set. In this new kind of problems, which we call multiclass and multiobjective equilibrium problems, the above mentioned characteristic feature arises: an equilibrium principle corresponding to the Wardrop's equilibrium principle is not uniquely defined. Moreover, even if we choose a particular kind of equilibrium principle, the equivalent formulation by means of a variational inequality is not ensured.

The paper is organized as follows. In Sect. 2 we consider two models (see [8,9]) in which two different kinds of Wardrop-type vector equilibria are proposed, and analyse and compare their VVI formulations. In Sect. 3 we propose a new VVI model which is equivalent to a suitable chosen Wardrop-type equilibrium principle (see Theorem 3.1) and in Sect. 4 we highlight our results by means of a pedagogical example. We also warn the reader that throughout the article we use the terms *multiobjective* and *multiclass* but the equivalent terms *multicriteria* and *multicommodity* are also used by other authors, particularly in economical applications.

2 The models

Let us first introduce the notation commonly used to state the standard traffic equilibrium problem, in the scalar case (see for instance [2,9]).

A traffic network consists of a set W of origin-destination pairs and a set $\mathcal{R}$ of routes. The set of all $r \in \mathcal{R}$ which link a given $w \in W$ is denoted by $\mathcal{R}(w)$. In our analysis we are not interested on the link structure of the routes. A route-flow vector is an element $F \in \mathbb{R}^{|\mathcal{R}|}$. Feasible flows are flows which satisfy the capacity constraints and demands, i.e., which belong to the set:

$$\mathbb{K} := \{F \in \mathbb{R}^{|\mathcal{R}|} : \lambda \leq F \leq \mu, \phi F = \rho\},$$

where $\lambda \leq \mu$ and ρ are given, and ϕ is the pair-route incidence matrix whose elements $(\phi)_{w,r}$ are set equal 1 if route r connects the pair w, 0 else. A mapping $C := \mathbb{K} \mapsto \mathbb{R}^{|\mathcal{R}|}$ is then given which associates to each flow $F \in \mathbb{K}$ its cost $C(F) \in \mathbb{R}^{|\mathcal{R}|}$.

Definition 2.1 A flow is called an equilibrium flow (or Wardrop Equilibrium) iff: $H \in \mathbb{K}$ and

$$\forall w \in W, \forall q, s \in \mathcal{R}(w), C_q(H) < C_s(H) \Longrightarrow H_q = \mu_q \text{ or } H_s = \lambda_s$$

that is equivalent to:

$$H \in \mathbb{K} \text{ and } \langle C(H), F - H \rangle \geq 0, \quad \forall F \in \mathbb{K}$$

Let us notice that we are considering capacity constrained networks (see e.g. Sect. 1 of Ref. [3]). Roughly speaking, the meaning of Wardrop Equilibrium is that the road users choose minimum cost paths, and the meaning of the cost is usually that of travel time. However, in many situations users behave according to more than one objective (i.e. travel time, travel cost, facilities on the chosen road), and this can be expressed by introducing, for each path, a vector cost whose components represent the various objectives in the choice of the path. Moreover, the flows through the network can be grouped in different classes. For instance, if we are dealing with an urban transportation network, it can be important to put in different classes public vehicles and private ones; if we are dealing with a multicommodity supply demand network, manufacturers and retailers may want to group in different classes the different commodities.

Now we shall introduce the model of the vector equilibrium problem introduced in Ref. [8]. We remark that the author in Ref. [8] deals, actually, with a multiobjective multiclass problem although the abstract functional setting used can somewhat hyde the analogies with the more concrete problem studied in Ref. [1].

Thus, Let X, Y, Z be three topological vector spaces. If X is the space of flows(for a given path), $X^{|\mathcal{R}|}$ represents the space of flows on the network, Z is the space of costs and Y is the space of linear mappings from X to Z. By $C[F]$ we understand the value of $C \in Y$ applied to $F \in X$. Let P_X,P_Y,P_Z, three (partial) ordering convex nonvoid cones of X, Y, Z, respectively. We also assume that P_Y is pointed, i.e. $P_Y \bigcap(-P_Y) = \{0\}$, and that $int\, P_X \neq \emptyset$. Now let us define the feasible set of flows:

$$K := \left\{ F \in X^{|\mathcal{R}|} : F_r \in \lambda_r + P_X, F_r \in \mu_r - P_X, \sum_{r \in \mathcal{R}} F_r = \rho_w, \forall r \in \mathcal{R}, \quad \forall w \in W \right\},$$

where λ_r, μ_r, ρ_w are given in X, and K is supposed nonempty. A map $C_r(\cdot) : K \mapsto Y$ is then assigned $\forall r \in \mathcal{R}$, which represents the vector cost function. At this point a *vector variational equilibrium* is defined as a solution of the following problem:

$$\text{Find } H \in K : \sum_{r \in \mathcal{R}} C_r(H)[F_r - H_r] \in P_Z, \quad \forall F \in K \tag{1}$$

Let us notice that the notation $\sum_{r\in\mathcal{R}} C_r(H)[F_r - H_r] \geq_{P_Z} 0$ is also used instead of (1) by some authors (see e.g. [7]). Although this inequality formulation is more direct than (1) and allows to see the l.h.s. as a directional derivative when it is the case, we shall keep the set theoretic notation (1) to make it easy the comparison with [8].

In order to see the connection between (1) and a Wardrop-type condition two assumptions are made in Ref. [8]:

$$\text{if } F \in P_X, \text{ and } C \in P_Y \text{ then } C[F] \in P_Z \tag{2}$$

$$\text{if } F \in -int\, P_X, \text{ and } C \in P_Y \setminus \{0\} \text{ then } C[F] \notin P_Z \tag{3}$$

In the finite dimensional case we shall deal with in the sequel, condition (2) can be interpreted as the request to have nonnegative cost values on nonnegative flows, while condition (3) means that, for negative flows, the associated cost cannot be positive. Although negative flows are meaningless in the traffic problem, the last technical assumption is needed to prove proposition (4.1) of Ref. [8] which we recall and rename in this note as:

Proposition 2.1 *Assume that* (3) *holds and let* $C = (C_r)_{r\in\mathcal{R}} \in Y^{|\mathcal{R}|}$. *The following condition is* (*only*) *necessary for* $H \in K$ *being a vector variational equilibrium*:

$$\forall w \in W, \quad \forall t, s \in \mathcal{R}(w) \text{ there holds:}$$

$$C_t(H) - C_s(H) \in -(P_Y \setminus \{0\}) \Rightarrow H_t - \mu_t \notin -int P_X \text{ or } H_s - \lambda_s \notin -int P_X \tag{4}$$

The condition expressed by (4) is a kind of Wardrop equilibrium condition. In order to analyze its meaning in concrete application and to compare this approach to that of Ref. [1], we specialize our functional setting. Let $X = \mathbb{R}^q$ represent the space of multiclass flows, i.e. there are q commodities to be distributed through the network. Let $Z = \mathbb{R}^p$ be the space of costs, i.e. there are p objectives according to which the goods are shipped. Thus, Y is the space of linear mappings from $\mathbb{R}^q$ to $\mathbb{R}^p$. The cones which induce the partial order are: $\mathbb{R}^q_+$, $\mathbb{R}^p_+$ and $\mathbb{R}^{p\times q}_+$, respectively. The choice of these cones is the more natural from the point of view of the application. Now it is clear the meaning of Proposition 2.1, which we can interpres as a:

Weak Vector Wardrop Principle. *Let us fix arbitrarily an origin-destination pair,* w, *in the network and two paths* s *and* t *which connect* w. *If for each class and for each objective, the cost along* t *is less than or equal to the cost along* s *(but strictly less for at least one objective and one class) then,* **at least for one class**, *the flow on* t *is maximum or the flow on* s *is minimum. If the network flows are not capacity constrained, this means that at least for one class the flow on* s *is zero.*

This definition needs further comments. To simplify the discussion we shall consider the case without capacity constraints. Let us remark that condition (4) implies that only when, for a certain pair of paths (s, t), each element $[Ci]^j_t - C[i]^j_s$ of the matrix $C_t(H) - C_s(H)$ is positive, then at least one component of the vector flow (i.e. *at least for one class*), H_t is zero. However, for each fixed O-D pair, the commodity flows that vanish may depend on the path under consideration. We remind the reader that, for the given path indexed by t, $[Ci]^j_t$ represents the component i of the vector cost corresponding to the component (class) j of the vector flow. We stress that in this formulation the model requires that it is enough to optimize with respect to (only) one component of the flow.

Let us now introduce the definition of multiobjective, multiclass equilibrium according to Ref. [1]:

Definition 2.2 A vector flow $H \in K$ is an equilibrium iff $\forall w \in W$ and $\forall s, t \in \mathcal{R}(w)$

$$C_s(H) - C_t(H) \in \mathbb{R}^{p\times q}_+ \setminus \{0\} \rightarrow H_s = 0 \tag{5}$$

Thus, we can state the:

Strong Vector Wardrop Principle. *If for each class and for each objective, the cost along* s *is greater than or equal to the cost along* t *(but strictly greater for at least one objective and one class), then the flow of* **all classes** *along* s *is zero.*

It is then clear the difference with the previous definition.

Moreover, the following theorem is proved in Ref. [1]:

Proposition 2.2 *If* $H \in K$ *satisfies:*

$$\langle C(H), F - H\rangle \in \mathbb{R}^{p\times q\times q}_+ \quad \forall F \in K \tag{6}$$

then H *is an equilibrium according to* (5).

Here, $C(F) \in \mathbb{R}^{p\times q\times|\mathcal{R}|}$, while the flows on the network are represented by $\mathbb{R}^{q\times|\mathcal{R}|}$ matrices. Thus, we are faced with two different vector variational inequalities, (1) and (6) and each of them is connected with a certain notion of equilibrium. However, by direct inspection one can realize that if $H \in K$ is a solution of (6), then it solves (1). In fact, (6) requires that, *for each objective*, the sum over r of the unitary cost for each class α times the difference $F_r^\beta - H_r^\beta$ be positive, allowing that α and β vary independently, which is a stronger requirement with respect to (1) (see also the illustrative example of Sect. 3).

We introduce now a *new* variational inequality associated to the network problem which implies the Wardrop-type equilibrium defined in (4). Our starting point is the same as in [8], however we drop the technical assumption (3) and define a variational equilibrium as a solution to the following problem:

$$\text{find } H \in K : -\sum_{r\in\mathcal{R}} C_r(H)[F_r - H_r] \notin P_Z \quad \forall F \in K \tag{7}$$

We remark that the requirement that the l.h.s. does not belong to P_Z is different from (1), and is closer to Pareto's optimization point of view.

Proposition 2.3 *Let* $\mathrm{int}\, P_X \neq \phi$ *and assumption* (2) *holds. If* $H \in K$ *is a solution of* (7), *then* $H \in K$ *satisfies* (4).

Proof Assume that (4) is not satisfied. Hence, $\exists w \in W, s, t \in \mathcal{R}(w)$: $C_t(H) - C_s(H) \in -(P_Y \setminus \{0\})$, $H_t - \mu_t \in -\mathrm{int}\, P_X$, $H_s - \lambda_s \in -\mathrm{int}\, P_X$. Then, there exists $\delta \in \mathrm{int}\, P_X$: $H_t - \mu_t + \delta \in -P_X$, $H_s - \lambda_s - \delta \in P_X$. Let us define a feasible flow in the following way: $F_t = H_t + \delta$; $F_s = H_s - \delta$, $F_r = H_r$, $\forall r \neq t, s$. Then, by direct calculation, and exploiting the assumption (2) we get the contradiction that: $-\sum_{r\in\mathcal{R}} C_r(H)[F_r - H_r] = (C_s(H) - C_t(H))[\delta] \in P_Z$. □

Remark It is also clear now the relationship among the three variational inequalities introduced so far: (6)→(1)→(7). The last implication follows immediately from that fact that a vector in the positive cone does not belong to the negative cone.

3 An equivalence result

In this section we prove that an equilibrium condition can be expressed equivalently by a VVI.

Theorem 3.1 The variational equilibrium given by (1) is equivalent to the Wardrop-type equilibrium:

$$\forall i \in \{1, \ldots p\}, \forall w \in W, \forall s, t \in \mathcal{R}(w):$$

$$[Ci]_s^j(H) > [Ci]_t^j(H) \to H_s^j = 0, \quad \forall j = 1 \ldots q \tag{8}$$

Proof Let us write the feasible set in a decomposed form, i.e. splitting the constraints according to each class, and, for the sake of simplicity, without capacity constraints:

$$K := \left\{ F \in (\mathbb{R}^q)^{|\mathcal{R}|} : F_r^j \geq 0, \sum F_r^j = \rho^j \quad \forall j = 1 \ldots q, \quad \forall r \in \mathcal{R}, \forall w \in W \right\}$$

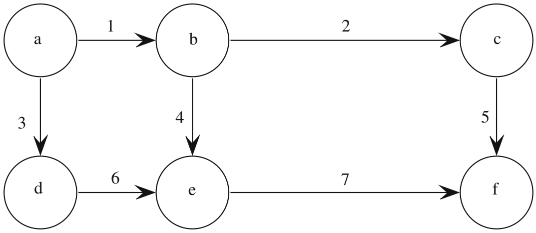

Fig. 1 Transportation network

Consider the VVI:

$$\sum_{r \in \mathcal{R}} C_r(H)[F_r - H_r] \in \mathbb{R}^p_+ \tag{9}$$

which represents a system of p scalar variational inequalities:

$$\sum_{r \in \mathcal{R}} \sum_{j=1}^{q} [Ci]^j_r(H)[F^j_r - H^j_r] \geq 0 \;\; \forall i = 1 \ldots p, \; \forall F \in K \tag{10}$$

Now, $\forall i = 1 \ldots p$ we can consider the corresponding variational inequality and choose $F_r = (H^1_r, \ldots, F^j_r, \ldots, H^q_r)$. Thus, each inequality in (10) implies a system of q inequalities:

$$\sum_{r \in \mathcal{R}} [Ci]^j_r(H)[F^j_r - H^j_r] \geq 0, \forall F^j : F^j_r \geq 0, \sum F^j_r = \rho^j \quad \forall j = 1 \ldots q, \forall r \in \mathcal{R} \tag{11}$$

On the other hand, if $H \in K$ satisfies (11), $\forall j = 1 \ldots q$, and fixed i, then $H \in K$ satisfies the ith inequality in (10). Thus we have proved the equivalence between the system of p inequalities in (10) and the system of $q \times p$ inequalities in (11). Now, each group of q inequalities obtained from (11) by fixing i, is *equivalent* to the classical Wardrop condition:

$$\forall w \in W, \forall s, t \in \mathcal{R}(w) : [Ci]^j_s(H) > [Ci]^j_t(H) \to H^j_s = 0, \quad \forall j = 1 \ldots q$$

Let us observe that, since we are assuming that a solution of (9) does exist, the p Wardrop conditions obtained by fixing i cannot be incompatible. □

4 An example

In this section we highlight, once more, the differences between the models previously described with the help of a simple example. The network, depicted in Fig.1, consists of six nodes {a,b,c,d,e,f} and seven arcs {1,2,3,4,5,6,7}. There are two types of commodities ($q = 2$), two objectives ($p = 2$) denoted by $C1$ and $C2$, and two O–D pairs. O–D pair $w_1 = (a, e)$ which is connected by the two paths: $R_1 = 1 \bigcup 4$, $R_2 = 3 \bigcup 6$ and O-D pair $w_2 = (b, f)$ which is also connected by two paths: $R_3 = 4 \bigcup 7$, $R_4 = 2 \bigcup 5$, so that $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$. The variational inequality to be associated to this network problem by using the formulation (6)— *Cheng and Wu's VVI*— is:

$$\langle C(H), F - H \rangle \in \mathbb{R_+}^{2\times 2\times 2}, \tag{12}$$

which is equivalent to a system of eight scalar variational inequalities which can be grouped in the following way:

$$\sum_{r\in\mathcal{R}}[C1]_r^{\alpha}(H)(F_r^{\beta}-H_r^{\beta})\geq 0 \tag{13}$$

$$\sum_{r\in\mathcal{R}}[C2]_r^{\alpha}(H)(F_r^{\beta}-H_r^{\beta})\geq 0 \tag{14}$$

$\forall\alpha,\beta=1,2,\forall F\in K$. Let us observe, that in this case, the (unitary) costs of a given commodity are multiplied for the flows of *each* class, so that it is not clear wether one can give a direct economical, or physical meaning to (12).

On the other hand, the application of (1)— *Oettli's VVI*— to this simple problem gives rise to the following system of two scalar variational inequalities:

$$\sum_{r\in\mathcal{R}}\sum_{h=1}^{2}[C1]_r^{h}(H)(F_r^{h}-H_r^{h})\geq 0 \tag{15}$$

$$\sum_{r\in\mathcal{R}}\sum_{h=1}^{2}[C2]_r^{h}(H)(F_r^{h}-H_r^{h})\geq 0 \tag{16}$$

Here, for each objective, the cost of each class is multiplied by the flow of the same class, and the quantities obtained are then summed up.

For the sake of completeness, let us observe that, the application of our model (7) to this example implies that, $\forall F\in K$, the vector:

$$\left(-\sum_{r\in\mathcal{R}}\sum_{h=1}^{2}[C1]_r^{h}(H)(F_r^{h}-H_r^{h}),\ -\sum_{r\in\mathcal{R}}\sum_{h=1}^{2}[C2]_r^{h}(H)(F_r^{h}-H_r^{h})\right)$$

does not belong to $\mathbb{R}_+^2$.

5 Conclusions

In this note we have shown that multiclass-multiobjective equilibrium problems can be modelled in different ways and can be associated to different kind of vector variational inequalities. We have compared the different notions of vector equilibrium existing in the literature, which are not equivalent to their associated vector variational inequalities, and shown that a formulation of the problem is possible which yields equivalence between a vector variational inequality and a suitably chosen vector Wardrop principle. Although most of the analysis has been performed in a finite dimensional framework for the sake of clarity, the extension of our results to infinite dimensional spaces requires only some technical modifications. In particular, when dealing with Lebesgue spaces, the hypothesis $int\,P_X\neq\emptyset$ has to be replaced by using the notion of quasi interior (see [5]).

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